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From social choice functions to dictatorial social welfare functions

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Abstract

A procedure to construct a social welfare function from a social choice function is suggested and it is shown that the dictatorial are the only unanimous social welfare functions that can be reconstructed from a social choice function that does not change the social choice when a defeated alternative is moved to the last position in all the individual preferences.

The difference between night and day can be illustrated by comparing the original version of this paper (night) with the final version that incorporates the two referees' suggestions and corrections (day). I am very grateful to them for taking the trouble to make this transformation (and recommending the publication of this paper) and to Jordi Massó, the associate editor, for accepting the publication of this paper.
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1. Introduction

Arrow's (1963) theorem, on the one hand, expresses an impossibility result: one cannot have a non-dictatorial preference aggregation rule (social welfare function) that satisfies unanimity and a condition of independence of irrelevant alternatives (IIA); see Feldman (1980) or Saari (1998, 2001). But, on the other, Arrow's theorem makes social choice theory challenging (and, therefore, interesting) because it merely makes apparent the inadequacy of a specific approach to construct reasonable social welfare functions.

For strict preferences, this note suggests the alternative approach consisting of constructing social welfare functions from social choice functions, which are rules transforming preference profiles not into preferences but into alternatives. The motivating idea is to define aggregation rules (social welfare functions) from simpler aggregation rules (social choice functions), so that one can use the solution to a certain aggregation problem to solve a more complex aggregation problem.

To see how just *one* social choice function can be used to construct a social welfare function, let P be a preference profile and g a social choice function defined on the universal domain and satisfying the weak Pareto principle[1] (see A4: if all individuals prefer alternative x to alternative y then g cannot select y). The first alternative in the social preference associated with P is $g(P)$. The problem is next to determine the second alternative in the social preference associated with P using g. One possible solution consists of replacing P with some preference profile P' that is "very close" to P and such that g does not associate with P' the previously selected alternative $g(P)$.

Since g satisfies the weak Pareto principle, moving $g(P)$ to the last position in each individual preference from profile P ensures that $g(P)$ will not be chosen. The resulting preference profile P' is in a sense "very close" to P, because the ordering of the "relevant" alternatives (all but $g(P)$) is the same in both profiles. So $g(P')$ could be chosen as the second alternative in the social preference associated with P.

The same logic could now be applied to P' to determine the third alternative in the social preference: just define P'' to be the preference profile obtained from P' by ranking

[1] I am very grateful to Referee 1 for calling to my attention that the iterative construction defined next is essentially the same one used in the proof of Lemma 2 in Ehlers and Weymark (2001) and to Referee 2 for noticing that, without the weak Pareto principle, the aggregation rule to be defined next need not be a social welfare function.

$g(P')$ last in each individual preference from preference profile P' and declare that $g(P'')$ is the third alternative in the social preference that corresponds to P. And so on.

The interest of this procedure is that the "instructions" needed to run a social welfare function could be "coded" by a social choice function, which is a structure simpler than a social welfare function. With always strict preferences, the main result presented in this note (Proposition 2.2) characterizes dictatorial social welfare functions by means of the weak Pareto principle (see A1: if all individuals prefer alternative x to alternative y then x is socially preferred to y) and the condition that, using the previous procedure, the social welfare function can be reconstructed from a social choice function that, in addition, satisfies the following property: if preference profile P' is obtained from P by ranking alternative $x \neq g(P)$ last in each individual preference then $g(P') = g(P)$[2]. In words, if all individuals give a defeated alternative the minimum support with everything else the same, then the winning alternative remains winning.

Since Proposition 2.2 is proved by showing that a social welfare function satisfying the preceding condition also satisfies IIA it follows that IIA can be subsumed under the approach of trying to reduce social welfare functions to social choice functions.

2. Definitions, assumptions and result

Let $N = \{1, 2, \dots, n\}$ be a finite set whose members designate individuals, A a finite set containing m elements representing alternatives and L the set of complete, transitive and asymmetric binary relations that can be defined on A. Each member of L expresses a strict preference relation on A (no two alternatives are indifferent) and can be represented by a vector $\pi = (x_1, \dots, x_m)$ of the m alternatives, with x_p preferred to x_q if, and only if, $p < q$. Intuitively, π consists of a ranking of m positions. For $1 \leq k \leq m$, let ${}^k\pi$ designate the alternative filling the kth position in ranking π.

For $\pi \in L$, $P \in L^n$, $x \in A$, $y \in A\backslash\{x\}$ and $i \in N$: (i) P_i denotes individual i's preference in preference profile P; (ii) $\pi|_{\{x,y\}}$ denotes the restriction of π to the set $\{x, y\}$; and (iii)

[2] Postulating the property that a social welfare function can be reconstructed from a social choice function makes the requirement that g satisfies the weak Pareto principle unnecessary, because the property presumes that the outcome of the procedure must be a social welfare function. It is this presumption that automatically restricts the domain of social choice functions that can be used to construct a social welfare function. Without this presumption, the domain restriction must be explicit; for instance, the social choice function must satisfy the weak Pareto principle.

$P|_{\{x,y\}}$ abbreviates $(P_1|_{\{x,y\}}, \ldots, P_n|_{\{x,y\}})$. The preference profile $P{\downarrow}x$ obtained from $P \in L^n$ by moving $x \in A$ to the last position in each P_i is the profile P' in which, for all $i \in N$, P'_i is defined as follows. Let $1 \leq k \leq m$ be such that $x = {}^kP_i$. Then ${}^mP'_i = x$ and, for all $1 \leq r < m$, ${}^rP'_i = {}^rP_i$ if $r < k$ and ${}^rP'_i = {}^{r-1}P_i$ if $r > k$.

Given N and A with $n \geq 2 < m$, a social welfare function is a mapping $f : L^n \longrightarrow L$. The interpretation is that, for $(P_1, \ldots, P_n) \in L^n$, $f(P_1, \ldots, P_n)$ is the social preference that f associates with individual preferences $P_1, \ldots, P_n$. Social welfare function f is dictatorial if there is an $i \in N$ such that, for all $(P_1, \ldots, P_n) \in L^n$, $f(P_1, \ldots, P_n) = P_i$; that is, individual i (called "dictator") is such that the social preference always coincides with i's preference. In this context, Arrow's (1963, p. 97) theorem asserts that, for $n \geq 2 < m$, every social welfare function that satisfies IIA (independence of irrelevant alternatives) and A1 (the weak Pareto principle) below is dictatorial.

IIA. For all $P \in L^n$, $Q \in L^n\backslash\{P\}$, $x \in A$ and $y \in A\backslash\{x\}$, if $P|_{\{x,y\}} = Q|_{\{x,y\}}$ then $f(P)|_{\{x,y\}} = f(Q)|_{\{x,y\}}$.

A1. For all $P \in L^n$, $x \in A$ and $y \in A\backslash\{x\}$, if xP_iy for all $i \in N$ then $xf(P)y$.

Suppose that, for all $i \in N$, x is ranked above y in P_i if, and only if, x is ranked above y in Q_i. By IIA, x is ranked above y in $f(P)$ if, and only if, x is ranked above y in $f(Q)$. Hence, the ranking between x and y at the social level depends only on the ranking between these two alternatives at the individual level. By A1, if x is above y in every individual ranking in P then x is above y in the social ranking $f(P)$.

Given N and A with $n \geq 2 < m$, a social choice function is a mapping $g : L^n \longrightarrow A$. The interpretation is that, for $(P_1, \ldots, P_n) \in L^n$, $g(P_1, \ldots, P_n)$ is the social choice that g derives from individual preferences $P_1, \ldots, P_n$. Social choice function g is dictatorial if there is an $i \in N$ such that, for all $(P_1, \ldots, P_n) \in L^n$, $g(P_1, \ldots, P_n) = {}^1P_i$. A2 next is the property that a social welfare function can be reconstructed from a social choice function as suggested in Section 1 when the social choice function is merely assumed to satisfy A3 (recall footnote 2). By A3, the social choice does not change when a defeated alternative is ranked last in all the individual preferences: if $Q \in L^n$ is obtained from P by ranking x last in every P_i then $g(Q) = g(P)$.

A2. There is a social choice function $g : L^n \longrightarrow A$ satisfying A3 such that, for all $P \in L^n$ and $1 \leq k \leq m$, ${}^kf(P) = g({}_kP)$, where ${}_1P = P$ and, for $2 \leq t \leq m$, ${}_tP = {}_{t-1}P{\downarrow}g({}_{t-1}P)$ (${}_tP$ is obtained from ${}_{t-1}P$ by ranking $g({}_{t-1}P)$ last in each individual ordering).

A3. For all $P \in L^n$ and $x \in A \setminus \{g(P)\}$, $g(P{\downarrow}x) = g(P)$.

Lemma 2.1. If $n \geq 2 < m$ and $f : L^n \longrightarrow L$ satisfies A2 then it satisfies IIA.

Proof. Suppose f does not satisfy IIA: there are $P \in L^n$, $Q \in L^n \setminus \{P\}$, $x \in A$ and $y \in A \setminus \{x\}$ such that $P|_{\{x,y\}} = Q|_{\{x,y\}}$, x is ranked above y in $f(P)$ but y is ranked above x in $f(Q)$. By A2, let g be the social choice function from which f is defined. Let $f(P) = (x_1, \ldots, x_s, x, y_1, \ldots, y_t, y, z_1, \ldots, z_u)$. Hence, by A2, ${}^1f(P{\downarrow}x_1) = g(P{\downarrow}x_1) = x_2$. Similarly, with $P' = P{\downarrow}x_1$, ${}^1f(P'{\downarrow}x_2) = g(P'{\downarrow}x_2) = x_3$. Thus, if $R \in L^n$ is obtained from P by ranking last in each P_i first x_1, then $x_2, \ldots$, and finally x_s then, by A2, ${}^1f(R) = g(R) = x$.

Since f satisfies A2, g satisfies A3 and $g(R) = x$, ${}^1f(R{\downarrow}y_1) = g(R{\downarrow}y_1) = g(R) = x$. In the same vein, with $R' = R{\downarrow}y_1$, ${}^1f(R'{\downarrow}y_2) = g(R'{\downarrow}y_2) = g(R{\downarrow}y_1) = x$. Accordingly, if $S \in L^n$ is obtained from R by ranking last in each R_i first y_1, then $y_2, \ldots$, and finally y_t and next first z_1, then $z_2, \ldots$, and finally z_u then, by A2, ${}^1f(S) = g(S)$ and, by A3, $g(S) = g(R) = x$. As a result, $S|_{\{x,y\}} = P|_{\{x,y\}}$ and, for all $i \in N$, the ranking of the last $m - 2$ alternatives in S_i is $(x_1, \ldots, x_s, y_1, \ldots, y_t, z_1, \ldots, z_u)$.

Let $f(Q) = (a_1, \ldots, a_p, y, b_1, \ldots, b_q, x, c_1, \ldots, c_r)$. An analogous argument proves that ${}^1f(T) = y$, where $T \in L^n$ is such that $T|_{\{x,y\}} = Q|_{\{x,y\}}$ and, for all $i \in N$, the ranking of the last $m - 2$ alternatives in T_i is $(a_1, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_r)$. Given that $\{a_1, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_r\} = \{x_1, \ldots, x_s, y_1, \ldots, y_t, z_1, \ldots, z_u\}$, the aim is now to rank the last $m - 2$ alternatives in each T_i in the same way as in each S_i without altering the socially most preferred alternative. To begin with, identify x_1 in $\{a_1, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_r\}$. By A3, $g(T{\downarrow}x_1) = g(T)$ so, by A2, ${}^1f(T{\downarrow}x_1) = {}^1f(T) = y$. This procedure can be applied to, in this order, $x_2, \ldots, x_s, y_1, \ldots, y_t, z_1, \ldots, z_u$ to define $V \in L^n$ from T such that: (i) by A2, ${}^1f(V) = y$; (ii) $V|_{\{x,y\}} = T|_{\{x,y\}}$; and (iii) for all $i \in N$, the ranking of the last $m - 2$ alternatives in V_i is $(x_1, \ldots, x_s, y_1, \ldots, y_t, z_1, \ldots, z_u)$. Given this and the fact that $P|_{\{x,y\}} = Q|_{\{x,y\}}$, $S|_{\{x,y\}} = P|_{\{x,y\}}$, $T|_{\{x,y\}} = Q|_{\{x,y\}}$ and $V|_{\{x,y\}} = T|_{\{x,y\}}$ imply $S|_{\{x,y\}} = V|_{\{x,y\}}$, it follows that $S = V$. Yet, ${}^1f(S) = x \neq y = {}^1f(V)$: contradiction.■

Proposition 2.2. If $n \geq 2 < m$ then $f : L^n \longrightarrow L$ is dictatorial if, and only if, it satisfies A1 and A2.

Proof. "$\Rightarrow$" If i is a dictator in f then: (i) A1 automatically holds; and (ii) A2 holds with $g : L^n \longrightarrow A$ such that, for all $P \in L^n$, $g(P) = P_i$. "$\Leftarrow$" If f satisfies A2 then, by Lemma 2.1, it satisfies IIA. As f satisfies IIA and A1, it is dictatorial by Arrow's theorem.■

A4. For all $P \in L^n$ and $x \in A$, if $g(P) = x$ then there is no $y \in A\backslash\{x\}$ such that, for all $i \in N$, yP_ix.

The credit of the following result should be attributed to Referee 2.

Proposition 2.3. If $n \geq 2 < m$ then $g : L^n \longrightarrow A$ is dictatorial if, and only if, it satisfies A3 and A4.

Proof. "⇒" It is easy to see that if g is dictatorial then A3 and A4 hold. "⇐" Suppose g satisfies A3 and A4 but it is not dictatorial. As Referee 2 observes, if g satisfies A4 then the social welfare function f defined from g as suggested in Section 1 is well-defined (because winning alternatives that have been moved to the last position will never be winning again) and satisfies A1. Hence, being A3 true for g, f satisfies A2 and, by Proposition 2.2, f has some dictator $i \in N$. If i is not a dictator in g then there is $P \in L^n$ with $g(P) \neq {}^1P_i$. By A2, ${}^1f(P) \neq {}^1P_i$, contradicting the fact that i is a dictator in f.■

Referee 2 suggests connecting Proposition 2.3 with the fact (Corollary 2.5) that social choice functions satisfying the weak Pareto principle (see A4) and Maskin (1999, p. 28) monotonicity (see A5) are dictatorial (see Muller and Satterthwaite (1977)).

A5. For all $P \in L^n$, $Q \in L^n\backslash\{P\}$ and $x \in A$, if $g(P) = x$ and, for all $i \in N$ and $y \in A\backslash\{x\}$, xP_iy implies xQ_iy then $g(Q) = x$.

Lemma 2.4. A5 is equivalent to A3.

Proof. "⇒" Let $g(P) = x$ and $z \in A\backslash\{x\}$. Then $Q = P{\downarrow}z$ is such that, for all $i \in N$ and $y \in A\backslash\{x\}$, xP_iy implies xQ_iy. By A5, $g(Q) = x$; that is, $g(P{\downarrow}z) = g(P)$. "⇐" Let $g(P) = x$. If, for all $i \in N$ and $y \in A\backslash\{x\}$, xP_iy implies xQ_iy then Q must come from P by exchanging contiguous alternatives (different from x) in the individual rankings. It therefore suffices to show that $g(Q) = g(P)$ when Q differs from P only in that, for some $i \in N$, $v \in A\backslash\{x\}$ and $z \in A\backslash\{v, x\}$, vP_iz and zQ_iv, where P and Q are such that, for no $t \in A$, vP_itP_iz and, for no $t \in A$, zQ_itQ_iv. Suppose $g(Q) \neq x$. Without loss of generality, assume $v \neq g(Q)$. By A3, $g(P{\downarrow}v) = g(P) = x$ and $g(Q{\downarrow}v) = g(Q) \neq x$. But $P{\downarrow}v = Q{\downarrow}v$: contradiction.■

Corollary 2.5. If $n \geq 2 < m$ then $g : L^n \longrightarrow A$ is dictatorial if, and only if, it satisfies A4 and A5.

Proof. Lemma 2.4 and Proposition 2.3.■

The preceding results provide additional links between Arrow's and the Gibbard (1973)-Satterthwaite (1975) theorem: the condition A3 is related to the condition IIA for social welfare functions through Lemma 2.1 and also to strategy-proofness of social choice functions, through Lemma 2.4 and the equivalence of A5 to strategy-proofness.

Referee 2 wonders whether Proposition 2.2 can be straightforwardly adapted to the case in which social indifference is allowed. In this case, since positions in the social preference can be filled by sets of alternatives and not just by singletons, social choice functions have to be replaced in A2 by social choice correspondences, which map preference profiles into sets of alternatives. Nevertheless, Lemma 2.1 does not hold for social choice correspondences. To see this, let g be the weak Pareto correspondence, where $g(P) = \{x \in A$: there is no $y \in A\backslash\{x\}$ such that, for all $i \in N$, $yP_i x\}$. With $N = \{1, 2\}$ and $A = \{x, y, z\}$, the profile P with $P_1 = (z, x, y)$ and $P_2 = (y, z, x)$ results in a social preference $f(P)$ with y indifferent to z and both preferred to x, whereas the profile Q with preferences $Q_1 = P_1$ and $Q_2 = (z, y, x)$ results in a social preference $f(Q)$ with z preferred to x and x indifferent to y. Thus, $P|_{\{x,y\}} = Q|_{\{x,y\}}$ but $f(P)|_{\{x,y\}} \neq f(Q)|_{\{x,y\}}$.

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